Beyond Morphology:
Uncovering Structure with the Wavelet Transform

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incorporating original work by:

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- A. Arneodo, F. Argoul, J-F Muzy, M. Tabard (CRPP, Bordeaux)
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- W. Sweldens (Lucent Technologies, Murray Hill)

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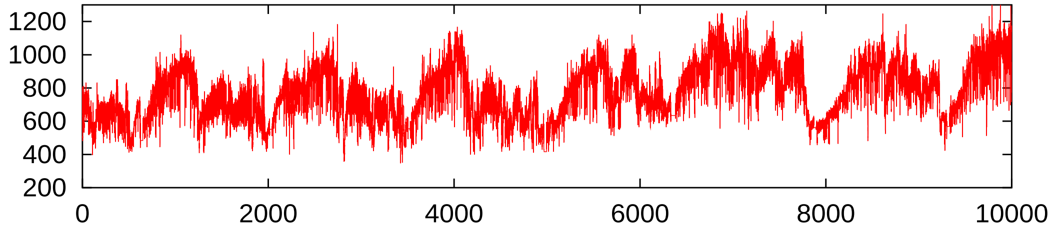
So what is it that <u>really</u> counts, structure or morphology?

It depends on your perspective...

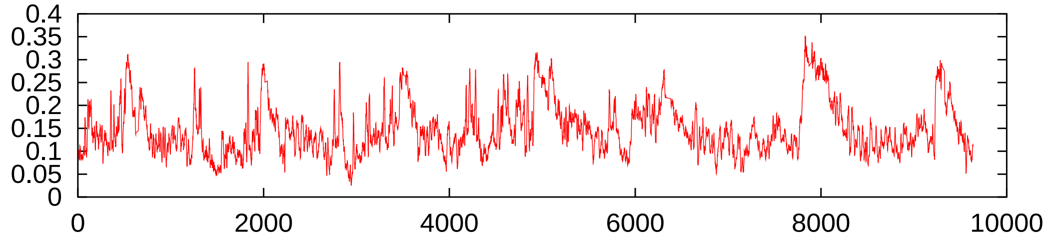
Contents

[1] You see the structure here...; with every point, I have less to say about it!

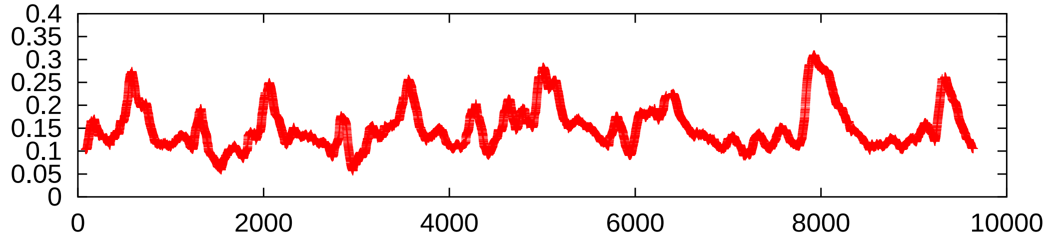
Is there any structure apparent in this time series?



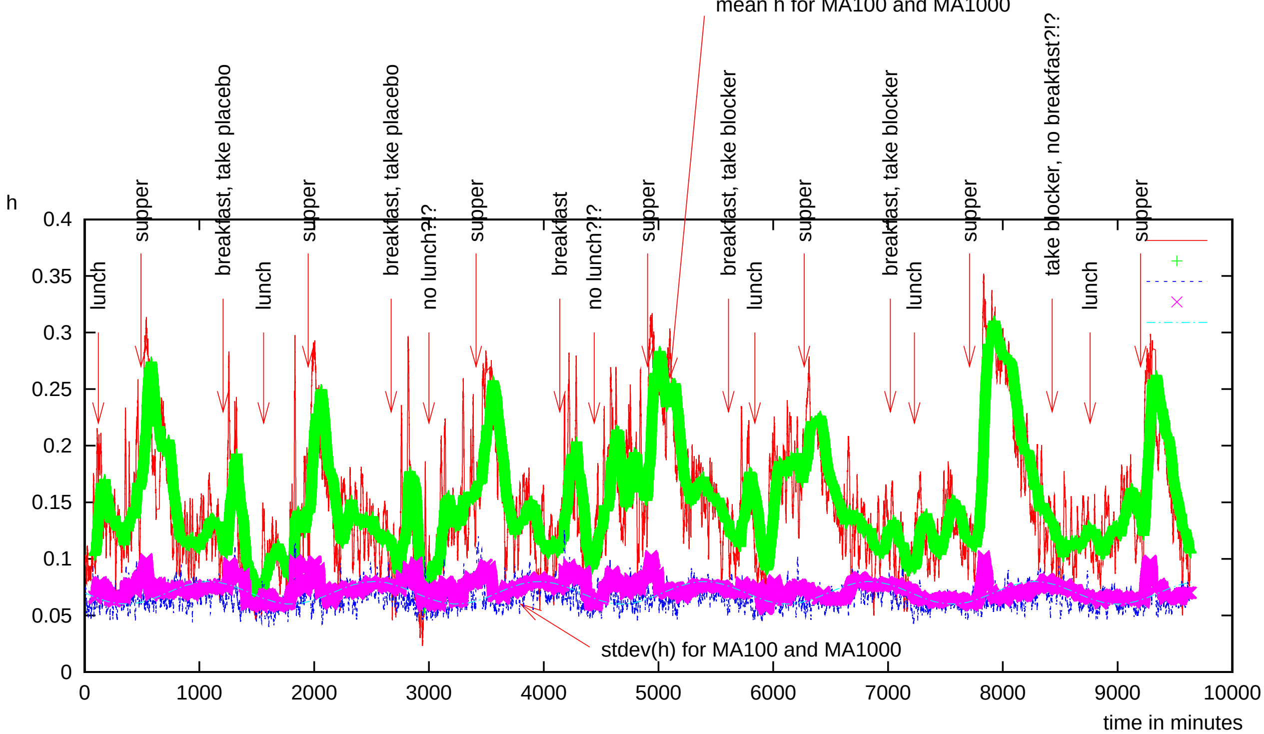
- let's look at the morphology first - local variability provides a good measure of morphological information/features



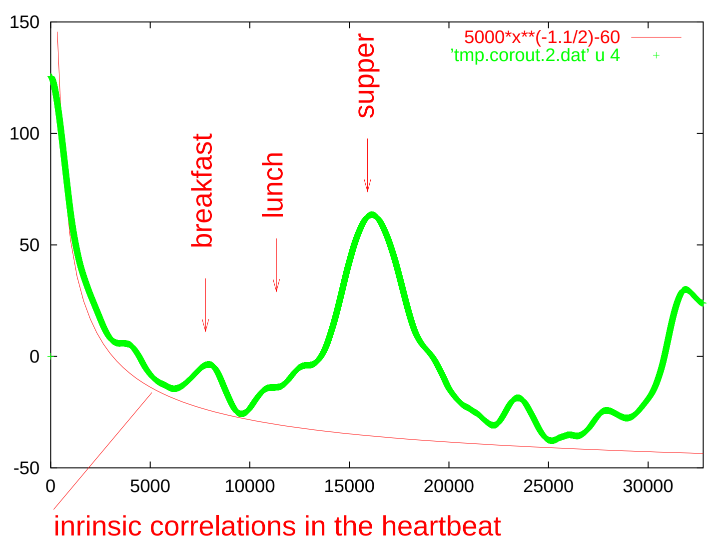
- for the estimate of local variability, we used the local effective Hölder exponent



- in order to reveal structure smoothing, MA kernels have been applied to enhance the collective properties of the variability



- Link to the external temporal information reveals structure!



- Autocorrelation function confirms the presence of invariant, intra-day periodic structure.

If our morphological description is:

- stationary
- isotropic
- uniform etc.

we are ready...

Then we can use its global characteristics (to represent the phenomenon:)

- distribution or
- power spectrum or
- fractal dimension or
- spectrum of dimensions

But if not (which is usually the case) we can look for the 'structure'

- **structure** $\equiv$ dynamical (time, position dependent) model of phenomena usually revealed by *correlations* in the raw data or in (morphological) features.
- Correlations can be just temporal or multiscale (across scales) or scale invariant (fractals!).

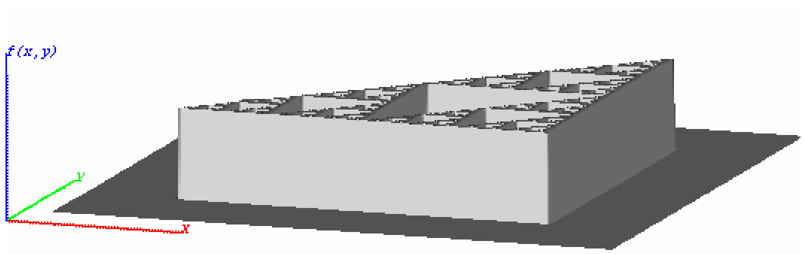
Examples of structure versus morphology
(*in increasing complexity*):

- The natural language
- DNA
- Supermarket customer[2]

[2] high correlation peak has been found for simultanous nappies and beer purchase just before the weekend!

Why Wavelets?

- To reveal morphological features at various resolutions
- To reveal the structure (of morphological features)



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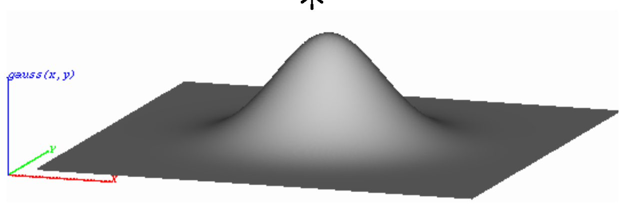

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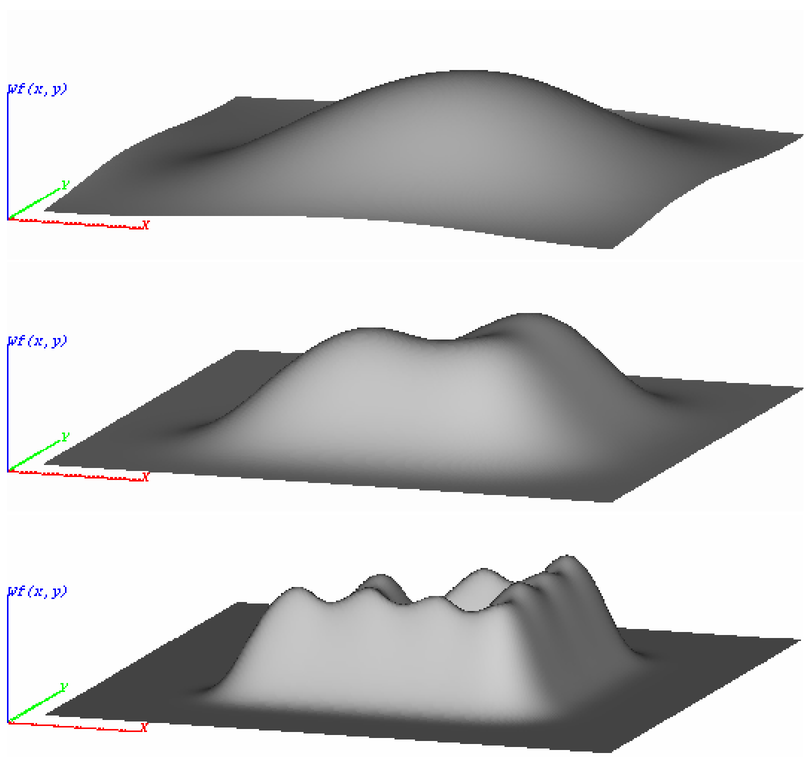
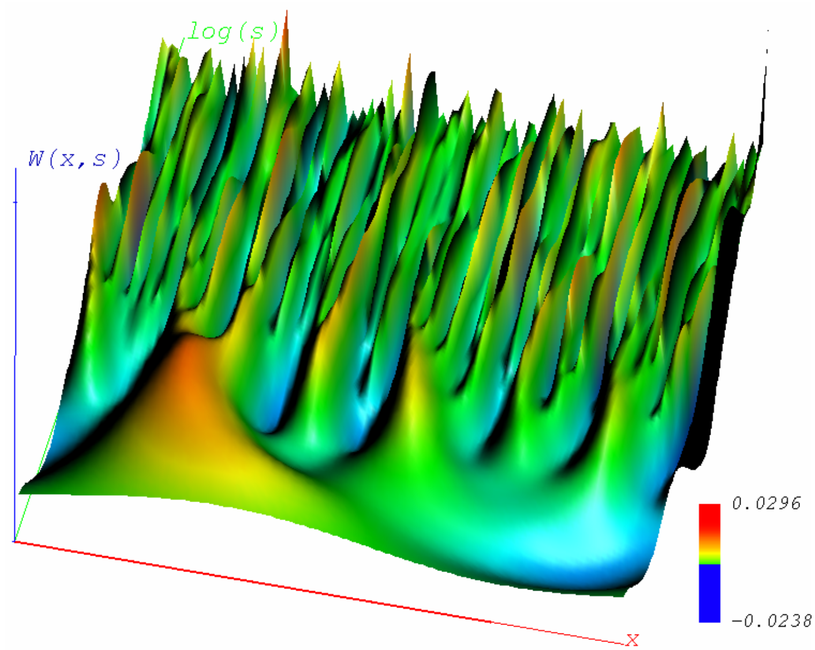


The wavelet transform is a convolution product of the signal with the scaled and translated kernel - the wavelet $\psi(x)$.

$$(Wf)(s,b) = \frac{1}{s} \int dx \; f(x) \; \psi(\frac{x-b}{s}) \; .$$



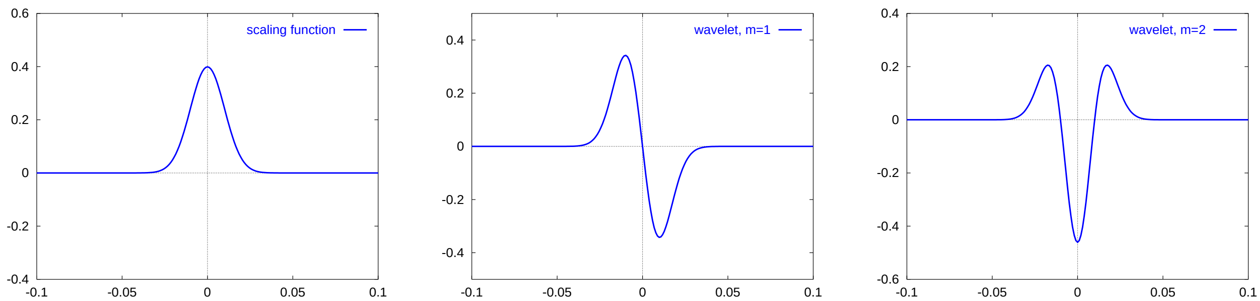
The scale parameter s 'adapts' the width of the wavelet kernel to the *microscopic resolution* required, thus changing its frequency contents; the location of the analysing wavelet is determined by the parameter b.

The Wavelet ψ

- The only requirement for the wavelet ψ is that the wavelet has zero mean - it is a wave function, hence the name *wavelet*.

- This *admissibity* requirement can be extended to orthogonality to polynomial of some degree n.

$$\int_{-\infty}^{\infty} x^n \, \psi(x) \, dx = 0$$



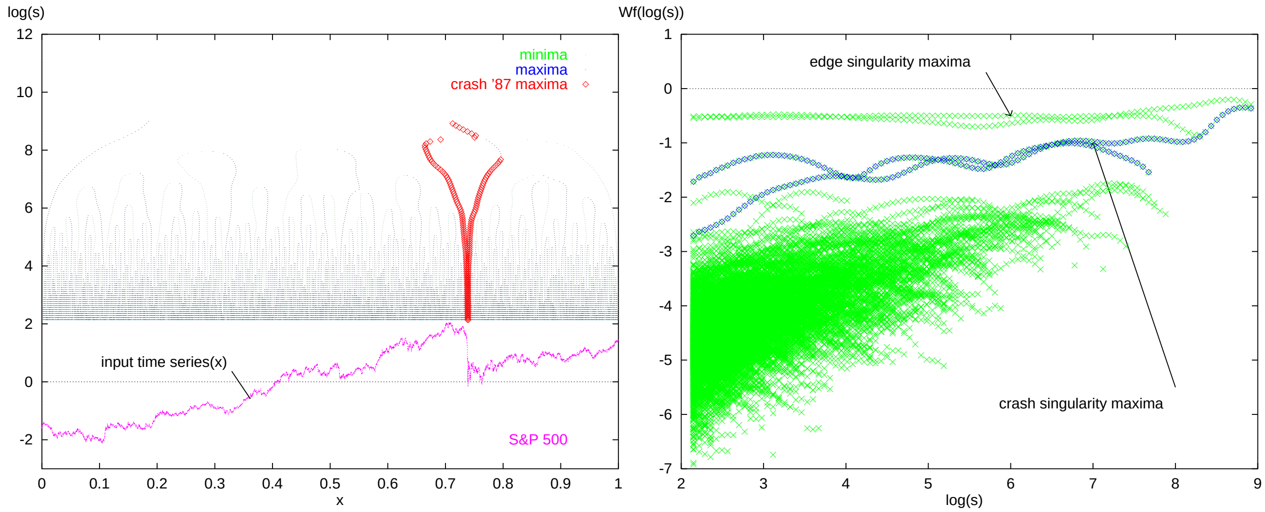
The Hölder Exponent

- The ability of the Wavelet Transform to filter polynomials of degree P_n is particularly useful for us since it allows the assessment of local scaling behaviour.
- This scaling behaviour is represented by the so-called Hölder exponent $h(x_0)$ of the function f(x):

$$|f(x) - P_n(x - x_0)| \leq C|x - x_0|^{h(x_0)}.$$

- We can be tempted to extract $h(x_0)$ from the scaling of the wavelet transform $Wf(x_0, s)$.

- Extracting the local scaling behaviour seems possible following the *modulus maxima* lines of the WT:



- The maxima converge towards the singularities.
- For *isolated* singularities of the *cusp* type, the Hölder exponent can be easily extracted.

Just as much as structure is a matter of 'model' discovery, morphology is a matter of definition!

It can be:

- a high frequency detail (e.g. an edge, or a set of)
- some low level model (e.g. the roughness exponent)
- a primitive shape (e.g. a circle, or its topological equivalent)

As such, morpology lends itself well to being characterised in terms of simple structures like wavelets.

Wavelets in Morphological Analysis

- local description
- in presence of bias/non-stationarity
- detailed information in terms of wavelets
  - generic wavelets (Mexican hat, Haar)
  - special wavelets (Cauchy, Morlet)
  - custom wavelets (lifting scheme of wavelet construction, Sweldens)
- we can see the wavelet as a 'morphological primitive'
- it can locally describe the data in terms of such a morphological primitive, providing characteristics like local exponent or instantaneous frequency - characteristic scale and location.

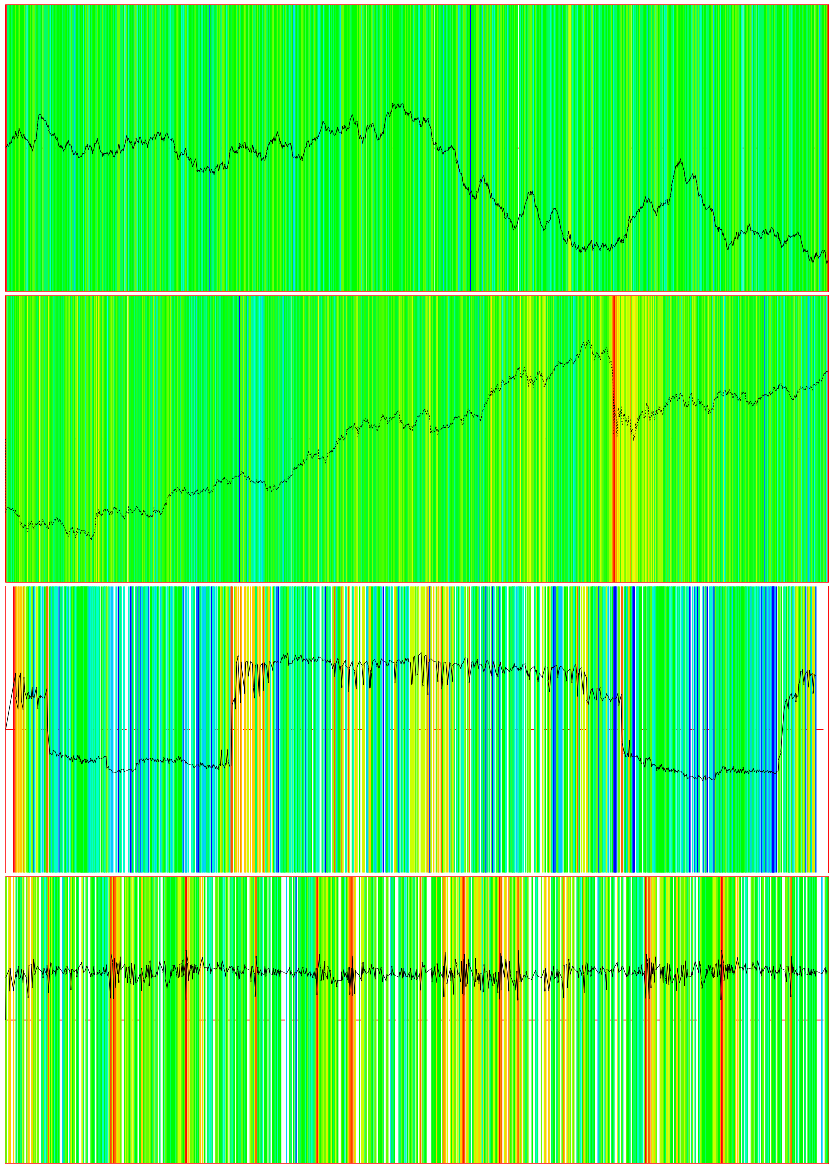
Wavelets in Structure Analysis:

- With the wavelet transform we can reveal :
  - hierarchy
  - nonstationarity
  - non-uniformity
  - non-isotropy

  of (morphological) features
- which can be used to search for a model

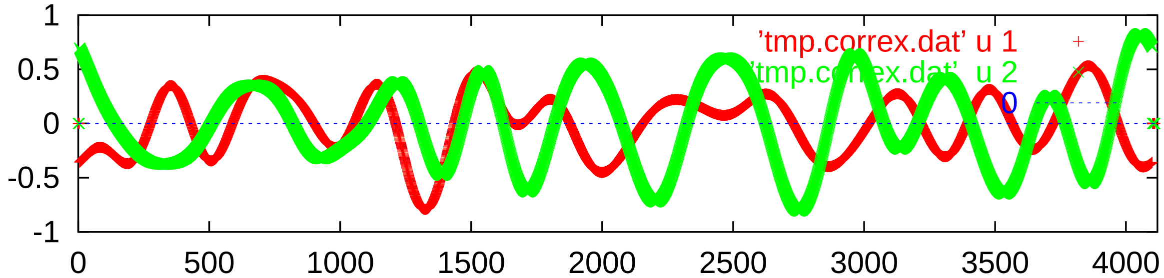
Structure can be seen as a departure from our simple (morpological) model.

- For example failure of the stationarity of the effective Hölder exponent:

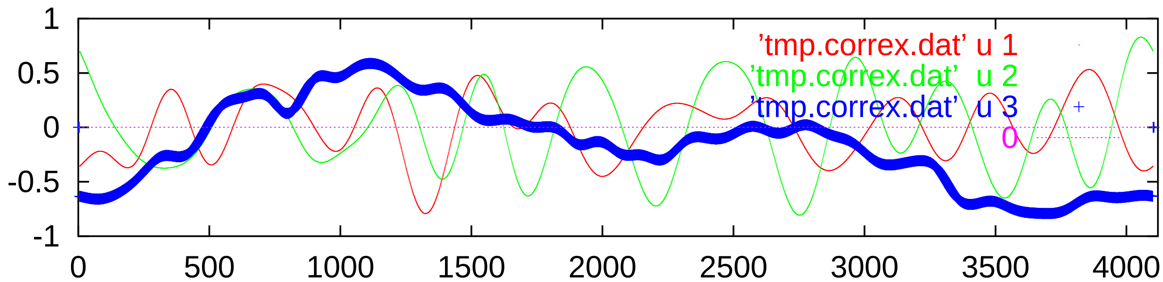


- But we need a way of detecting structure!

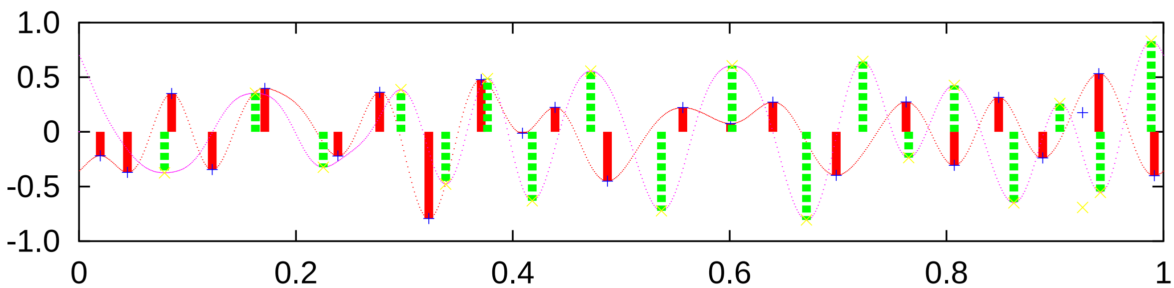
Recovering structure - it's all about correlations...



- direct correlation product (with normalisation)

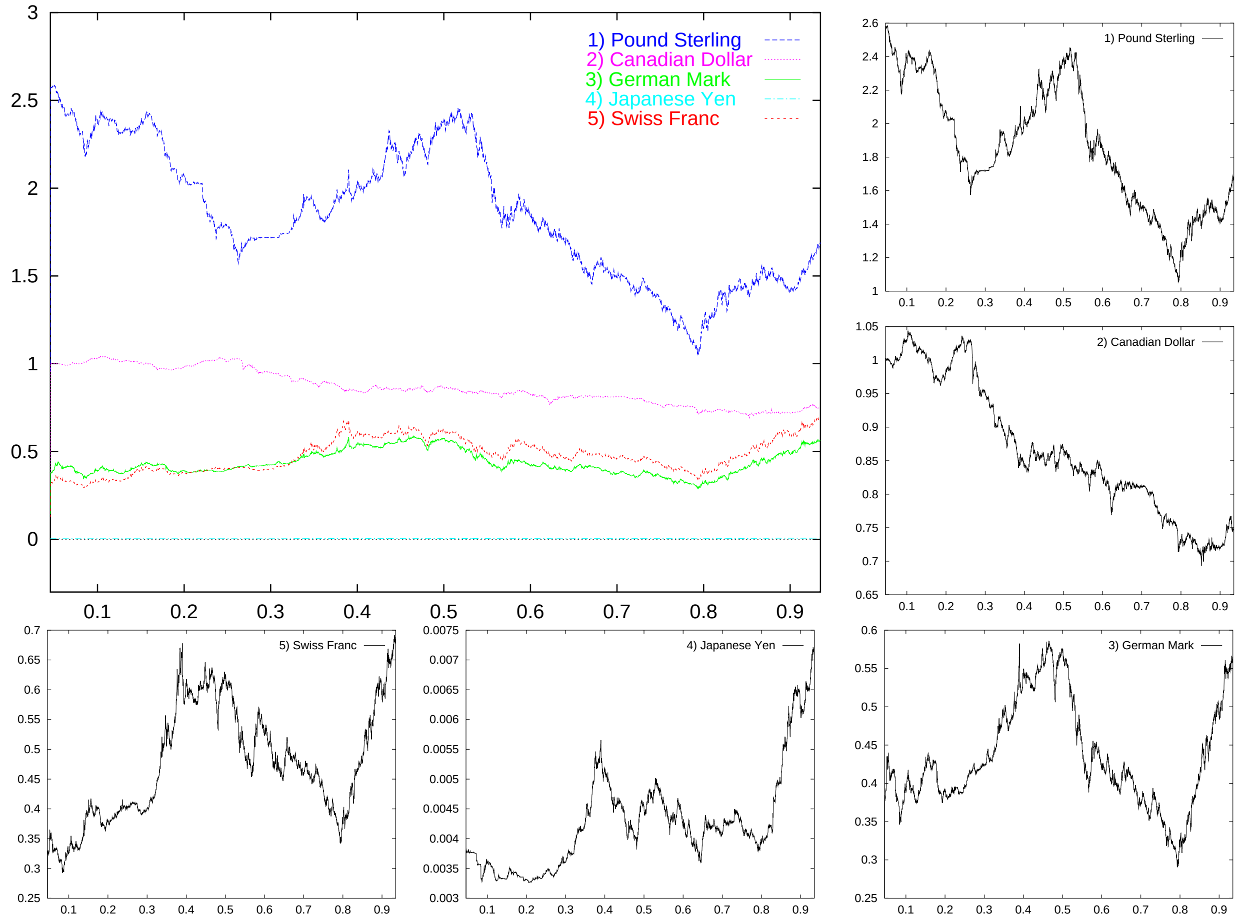


- ϵ neighbourood; *voting* - image processing community, Mallat for IFP



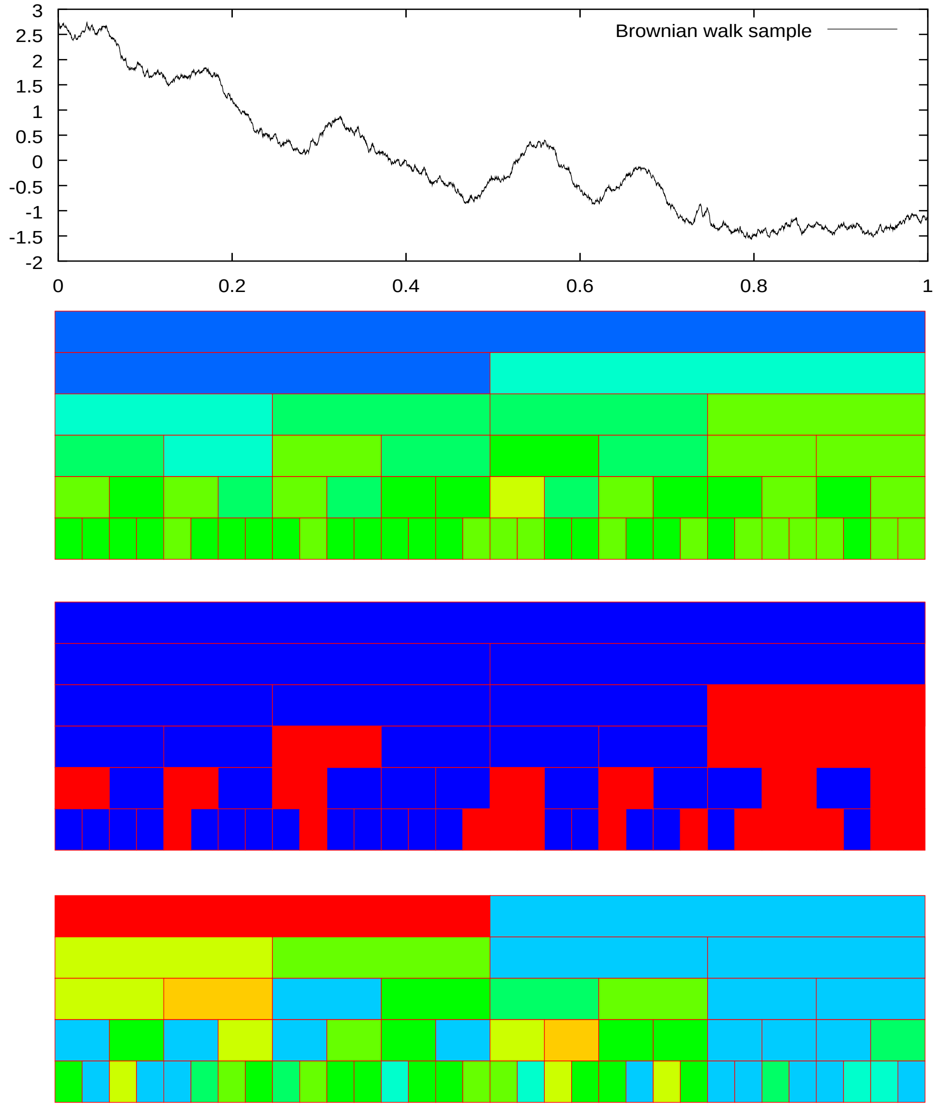
- match neighbouring maxima (and you may recover the mapping function!) Arneodo for IFP
- while matching it's good to optimize some *cost* function - dynamic time warping (speach recognition)
- and more...

But what if our morphology IS complicated and we don't really have a model for it?



- Problem: correlations between stocks or currencies.

The wavelet transform can still be used to represent morphological features!



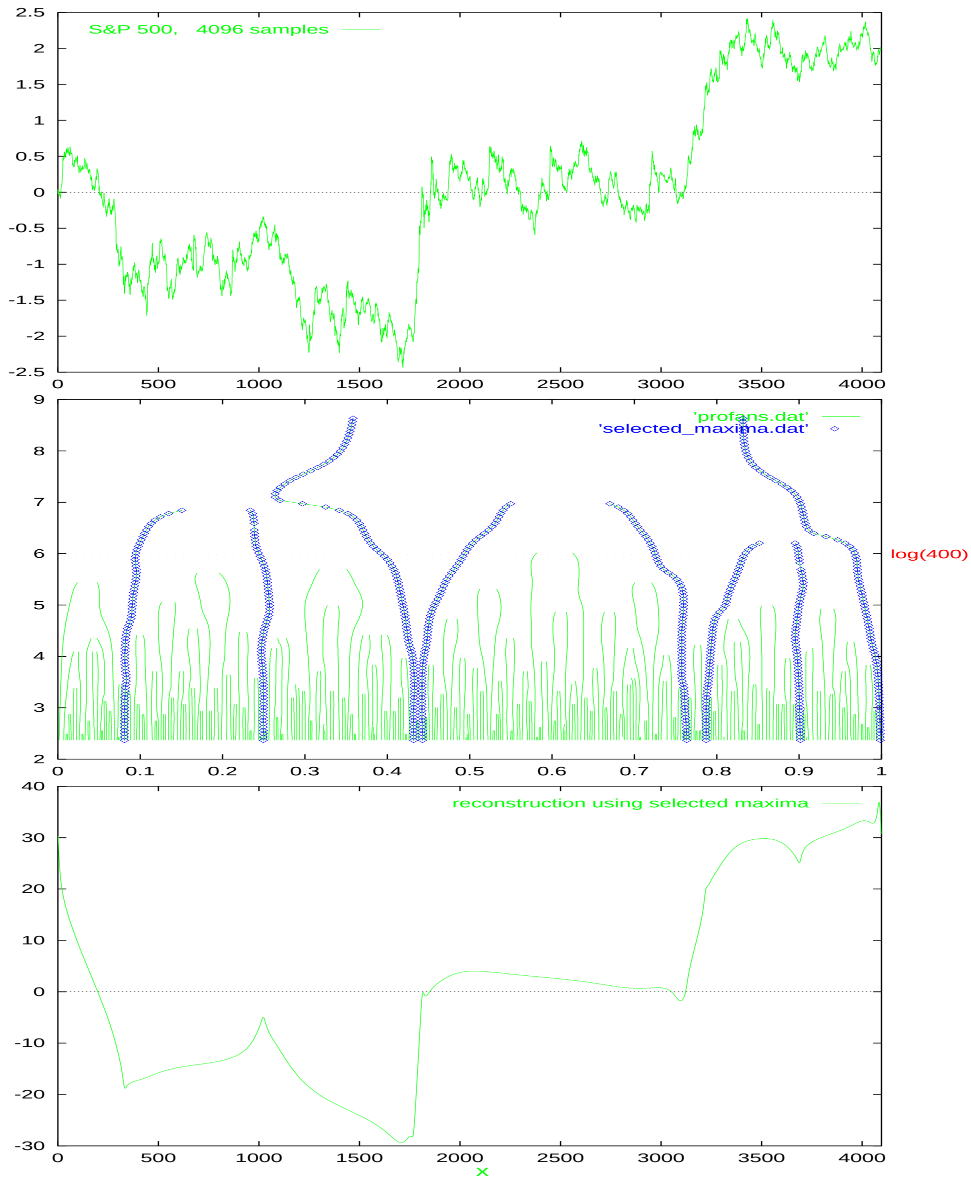
From above to below:

- input signal
- Haar WT - local average slope
- sign of Haar WT - slope up/down
- normalised log(amplitude) of Haar WT - local 'roughness' (red - rough, blue - smooth)

All these represent some aspects of morphology of the input signal and can be used for similarity matching with other time series.

h-Representation - another look at morphology with wavelets

- We can use the largest maxima of the CWT to partition the time series

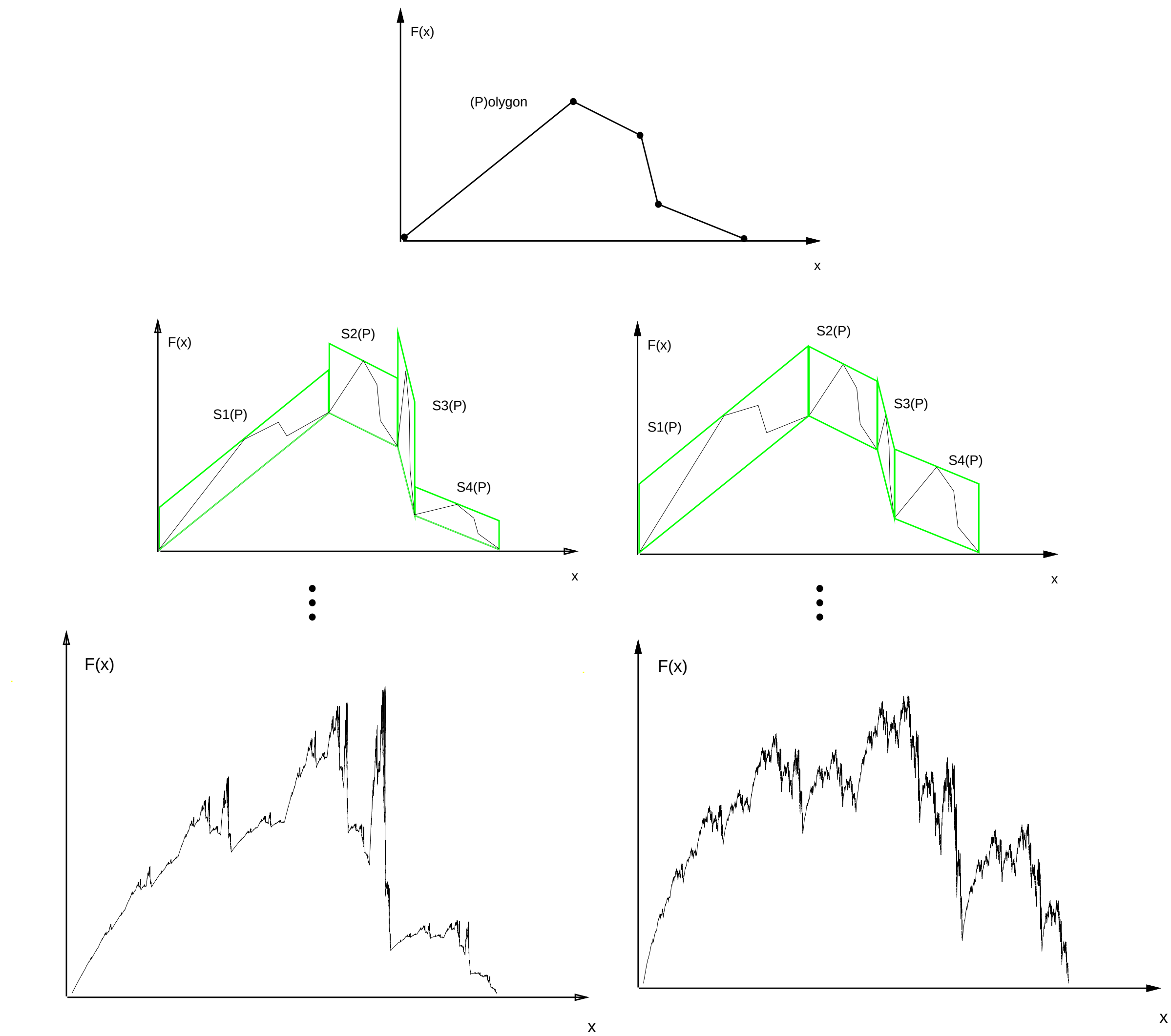


- Each point can be characterised by its local correlations exponent, or roughness exponent

h-representation = { position, Hölder exponent, sign }

Again, this 'morphological' representation can be used to search for structure through mutual correlations

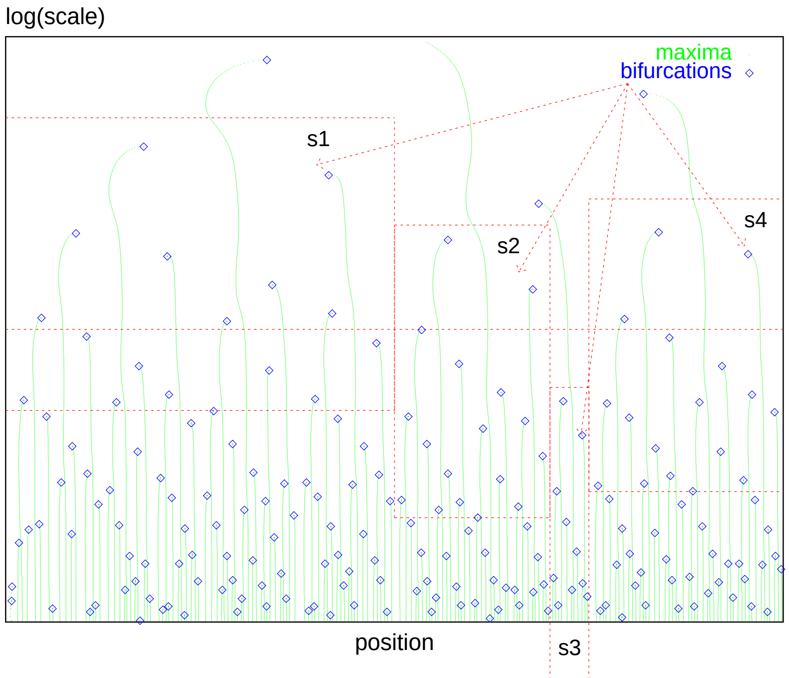
Fractals - example for scale invariant structures!

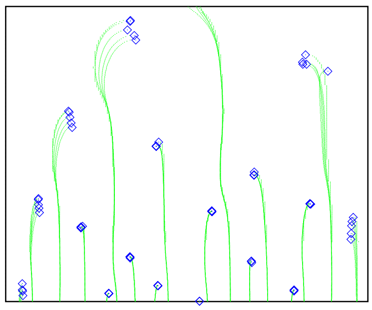


- self-affine IFS function construction
- by construction invariant with respect to some (contractive) affine operator

Generic formula for IFS:

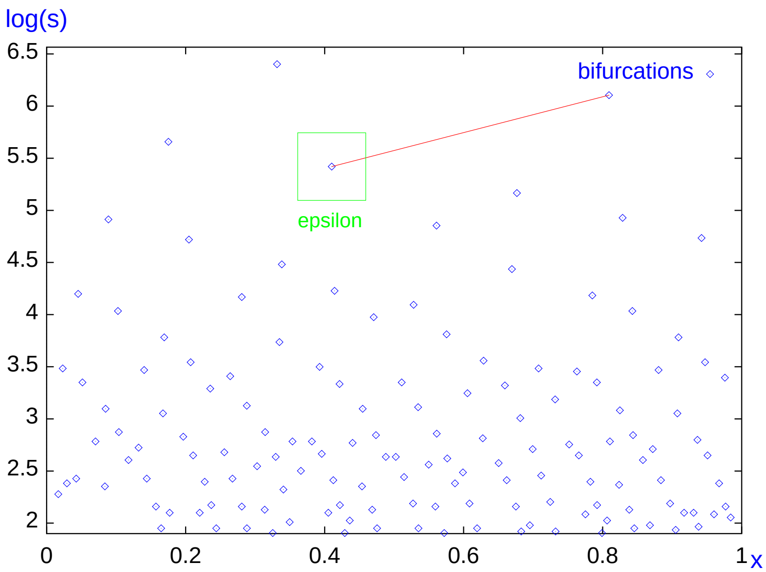
$$f'(x) = \alpha_n\, f(\frac{x-\beta_n}{\sigma_n}) + \gamma_n\,(x-\beta_n) + \delta_n$$

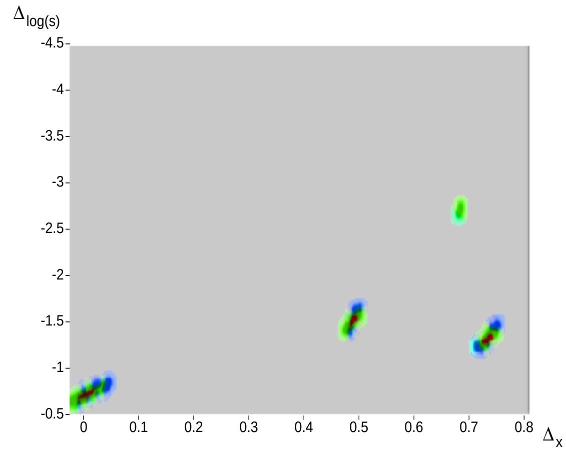




Affine invariance w.r.t. $(x-\beta_n)/\sigma_n$ revealed with the Mexican hat wavelet - second derivative of the Gaussian smoothing kernel.

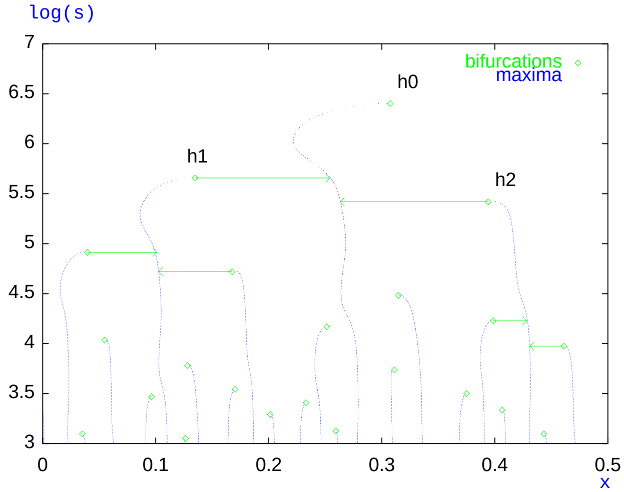
ϵ neighbourhood, voting scheme, Mallat et al 1993

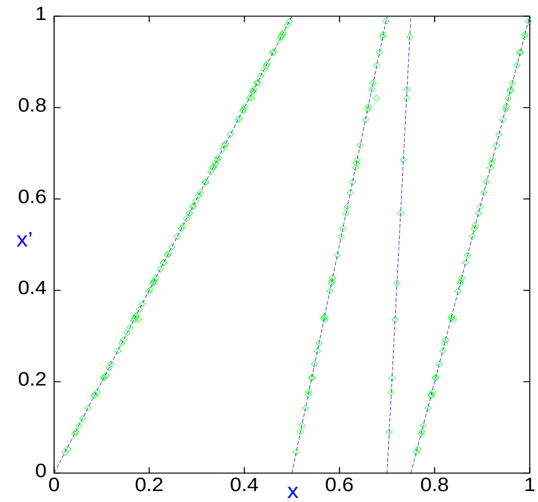




- voting density field V scanned with parameters $\Delta \log(s), \Delta x$.
- (morphological) features following the same scaling law contribute to the voting density field $V(\Delta \log(s), \Delta x)$.
- Limitation: only linear operations in x and $\log(s)$ can be recovered.

Tree matching scale invariance recovery, Arneodo et al. 1993



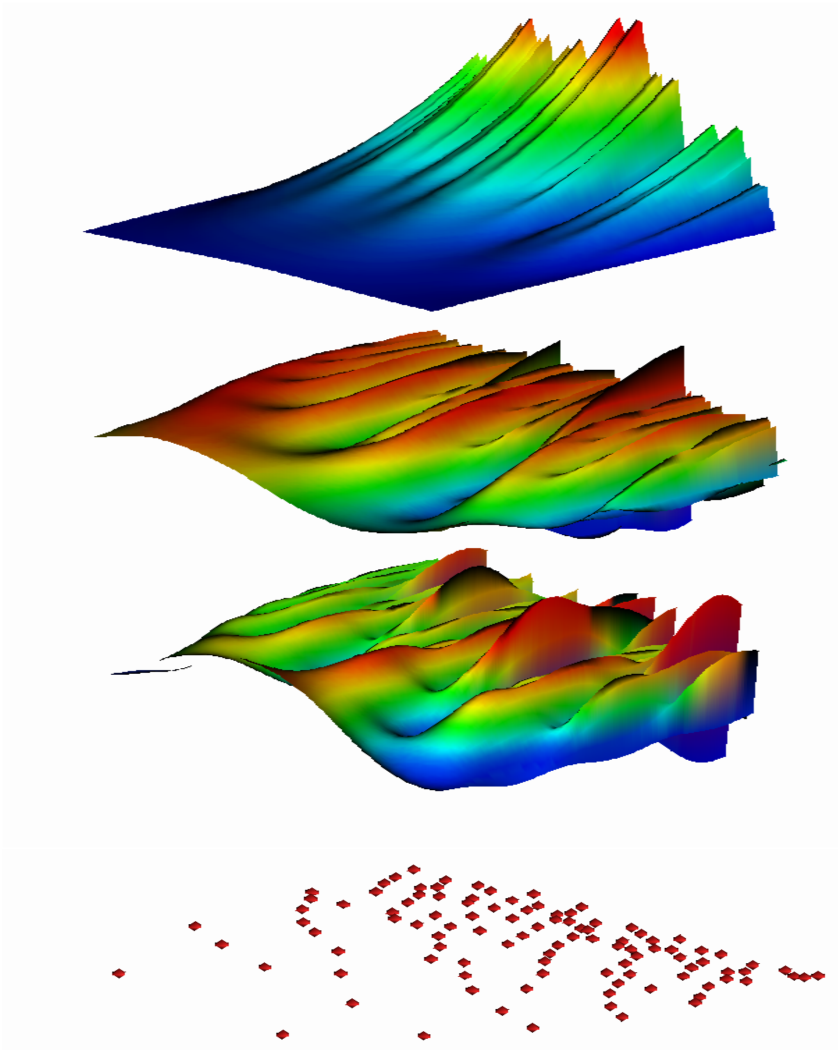


- dynamical iterative maps revealed!
- non-linear invertible transformations can be recovered!
- drawback: match optimization has to be performed

Using estimated parameters $\sigma_n, \beta_n, \alpha_n$, we can solve the set of equations

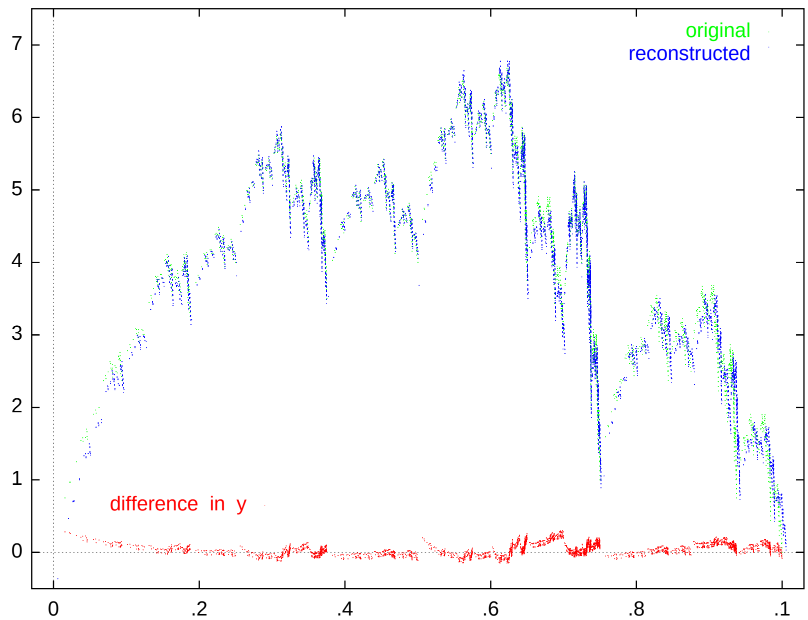
$$\left\{ \begin{array}{lcl} f'(x) & = & \alpha_n \, f(\frac{x-\beta_n}{\sigma_n}) + \gamma_n \, (x - \beta_n) + \delta_n \\ (D^{(1)} f')(x) & = & \alpha_n \, (D^{(1)} f)(\frac{x-\beta_n}{\sigma_n}) \, {\sigma_n}^{-1} + \gamma_n \\ (D^{(2)} f')(x) & = & \alpha_n \, (D^{(2)} f)(\frac{x-\beta_n}{\sigma_n}) \, {\sigma_n}^{-2} \end{array} \right. \tag{1}$$

by sampling the corresponding wavelet decompositions on the recovered invariant grid:



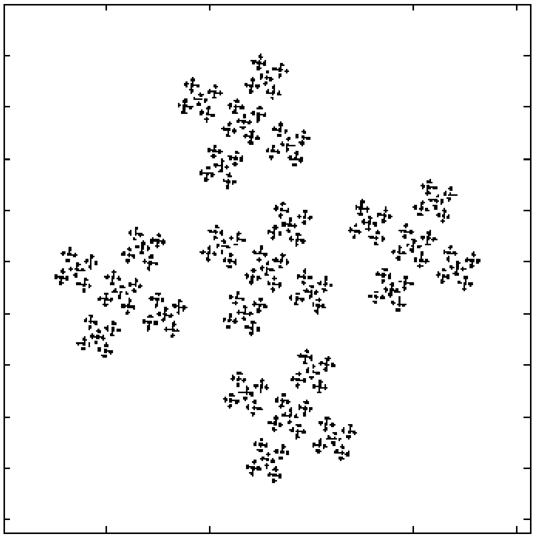
A stack of wavelet transforms $W^{(2)}f$, $W^{(1)}f$ and $W^{(0)}f$, sampled with the affine grid of bifurcations.

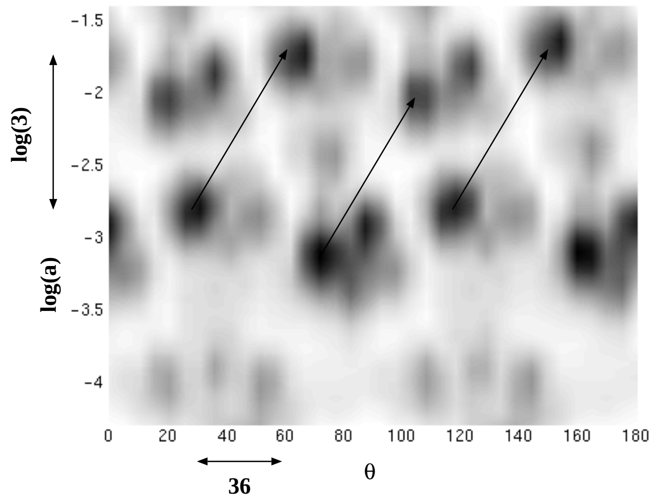
Once the IFS model is found the function can be reconstructed from it.



The original attractor, the attractor reconstructed and the reconstruction error

Extending structure detection to 2 dimensions





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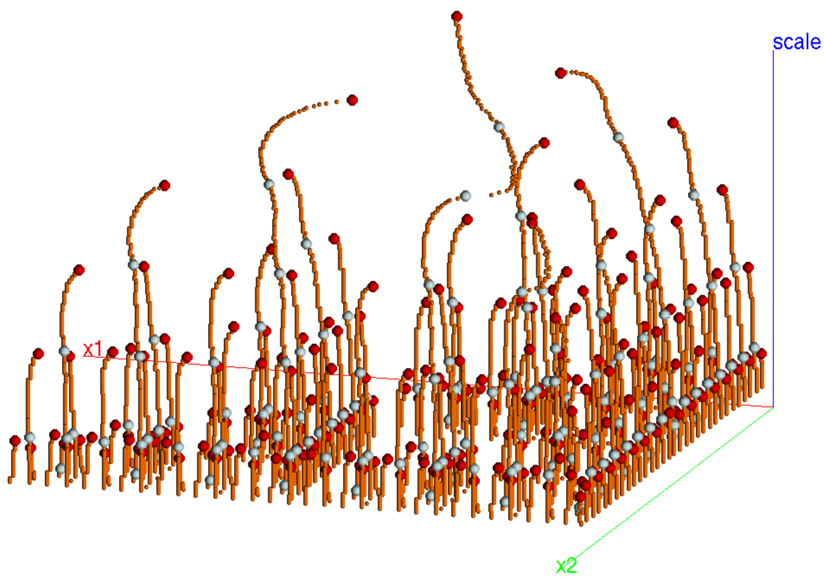
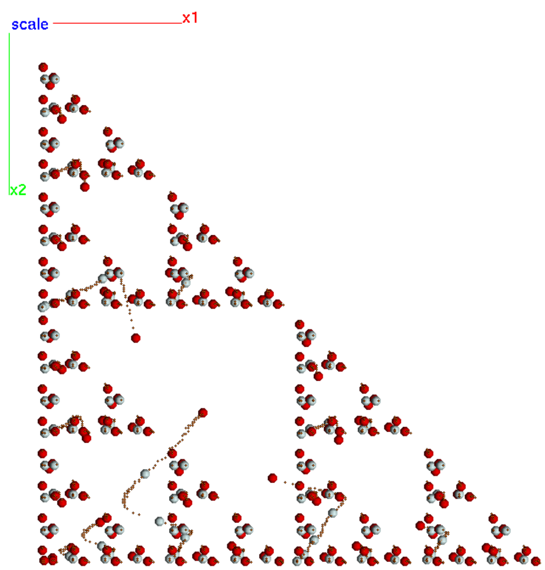
Twisted snowflake J-P. Antoine et al, *Proc SPIE*, 1999

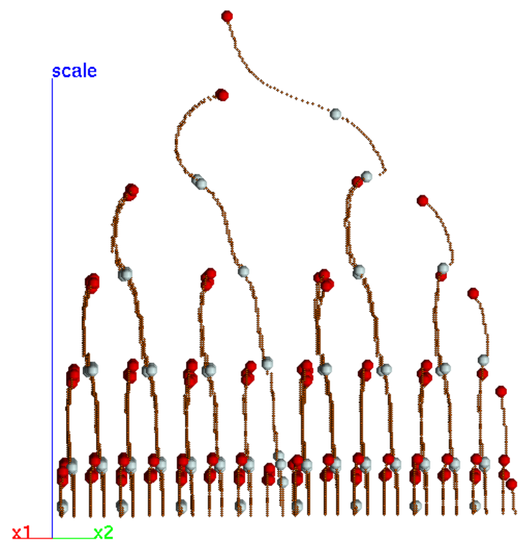
- voting can be used to show what are the symmetries in the pattern
- rotation invariance transformation can be revealed
- for scale invariant structures, scaling transformation can be revealed

Tree matching scale invariance recovery in 2-D

Complete WT maxima tree for the Sierpiński triangle

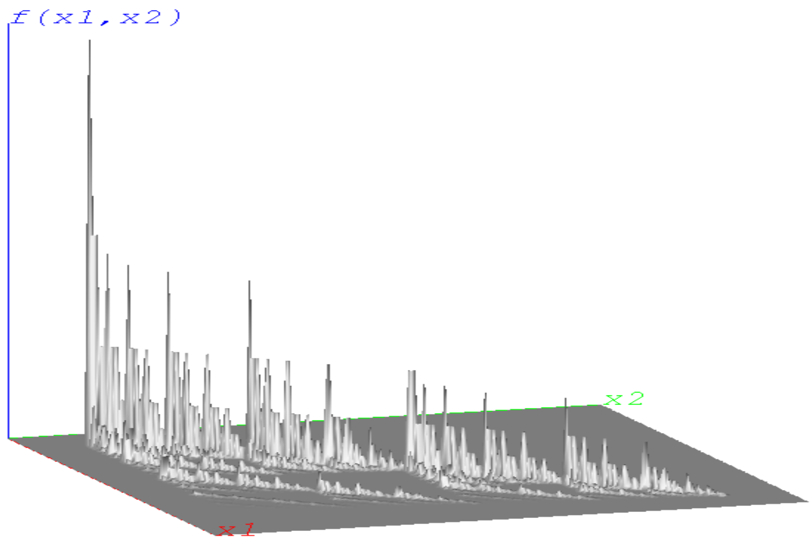




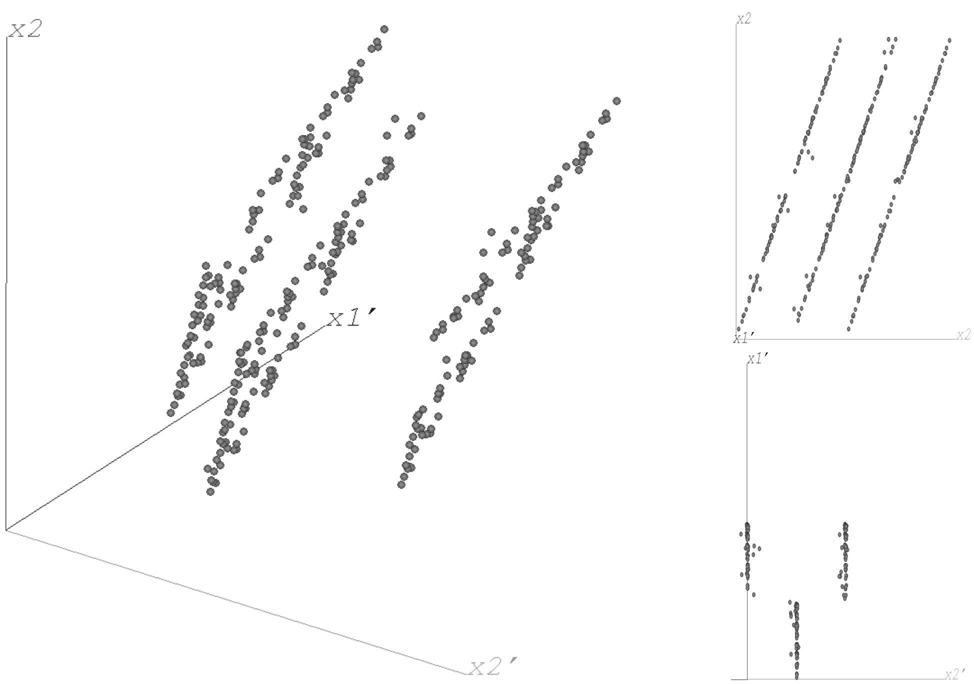


top view *and* side view

Multiplicative measure on the Sierpiński triangle

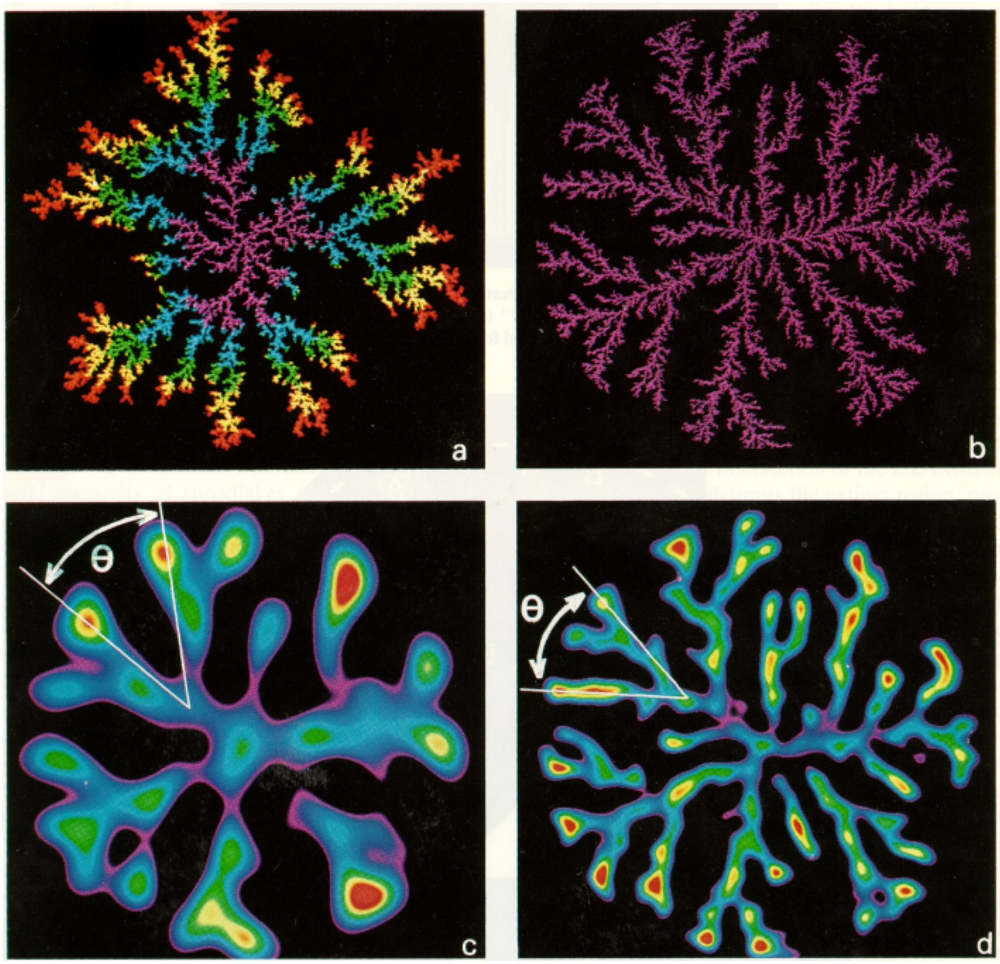


Recovered set of 3D maps:



Quasi-fractals are around us!

- Is there any structure to DLA clusters?



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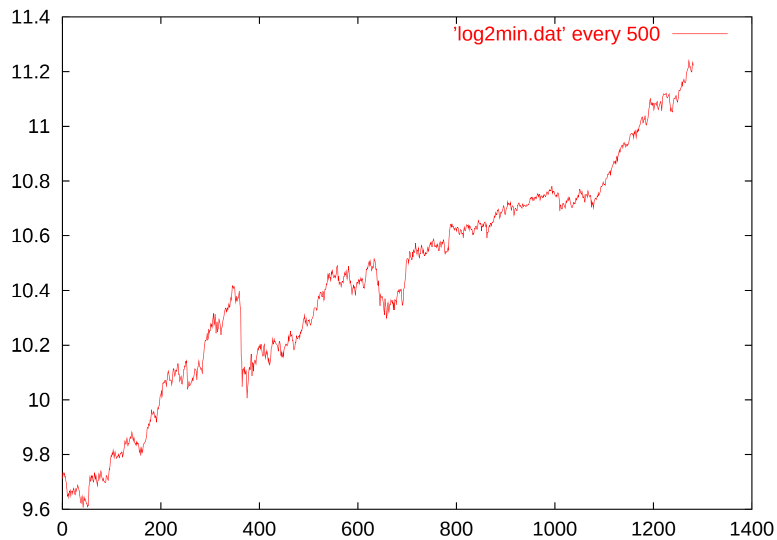
A. Arneodo et al, *Phys. Lett. A*, 1992

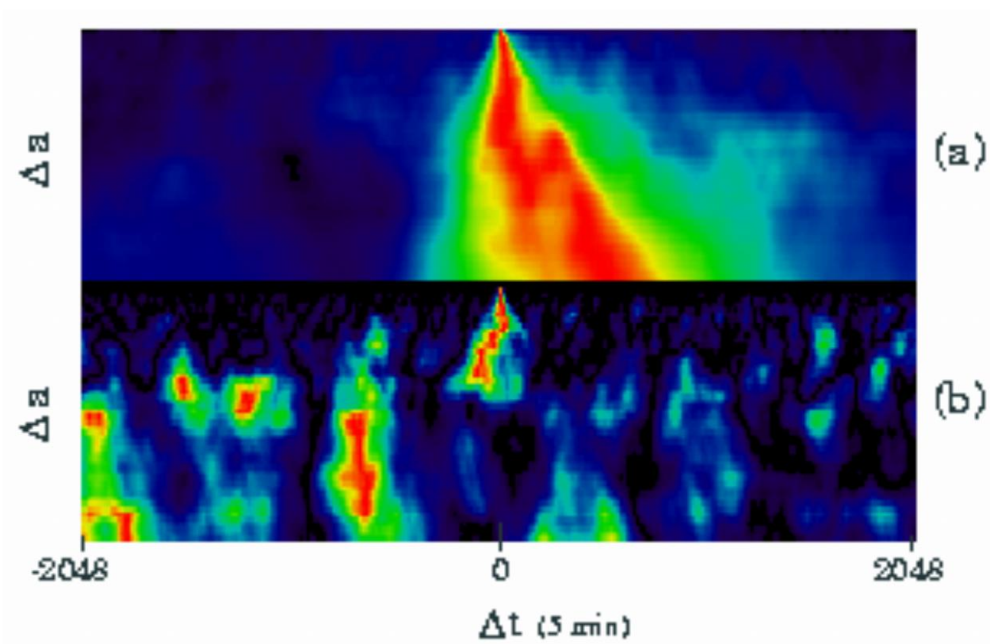
- five-fold symmetry
- Fibonacci branching

But how about a really complex phenomenon?

- Can we say anything about the 'structure' of the time series like S&P 500?



local log-volatility(a,t) $\rightarrow$ correlations$(\Delta a, \Delta t)$ $\rightarrow$ mean mutual information



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A. Arneodo et al. *Eur. Phys. J. B*, 1998

- The extraordinary fact: non-symmetric *propagation cone* of information! (upper picture, a))
- Shows that the volatility at large scales influences causally (in the future) the volatility at shorter scales.
- Surrogate test (lower picture, b)) - randomly shuffled version of the data

Σ-ing up...

Wavelets in Morphological Analysis

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- in presence of bias/non-stationarity
- detailed information in terms of wavelets
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